Understanding the Future

Daniel Research Group

The Mathematics of Growth and the Limits to Growth

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Stephen J. Daniel is founder and president of **Daniel Research Group,** a technology market research firm specializing in the design, development and application of market models and forecast. The following article contains material drawn from his book, *Understanding the Future, A Practical Guide to Designing and Developing Context Specific Segmented Forecasts and Models For Technology Markets*

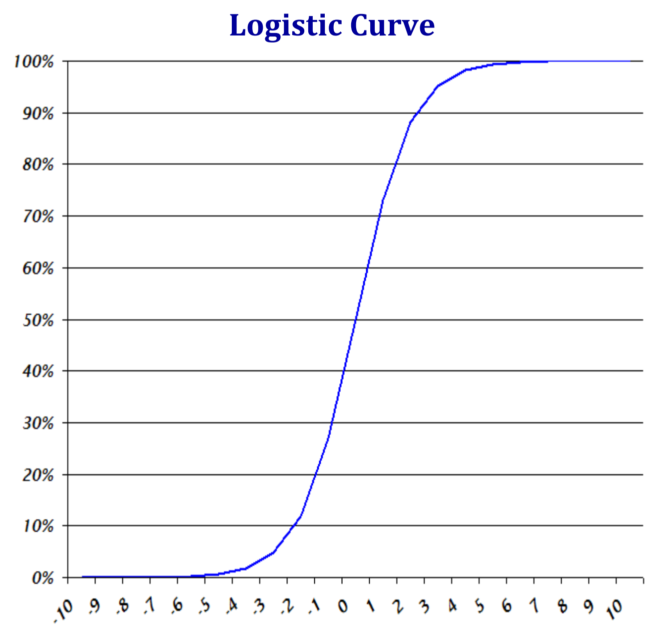
Given a heterogeneous population of potential adopters that is receiving information transmitted by internal and/or external communication channels, the observed state change (adoption) will follow an "S' shaped pattern that will exhibit accelerating growth at the start, transitioning to decelerating growth in the latter period. This can be represented in mathematical from by:

$$Y_t = \frac{1}{1 + f(t)}$$

Where *t* = a time period variable and $Y_t =$ the percent of the total population that has adopted at time *t*. From a mathematical point of view the value of Y_t will grow from almost zero to almost 1 as *t* goes from -∞ to +∞. Therefore $f(t)$ must go from a very large value to a very small value over the same time period. Additionally, since Y_t must exhibit the non-linear characteristics of accelerating growth transitioning to decelerating growth, $f(t)$ is best expressed as an exponential growth function of the form N^t, where *N* may theoretically be any number. However, in order to facilitate common mathematical operations is customary to let $N = e$, the natural logarithm constant. Furthermore since *t* will be progressing from negative values to positive values then the exponential function needs to be of the form e^{-t}.

The final equation then becomes:

$$Y_t = \frac{1}{1+e^{-t}}$$



This is the basic logistics function. In this form, it is not very useful since it has no parameters to control growth/shape or location. However, certain important properties can be discovered and explored using this form rather than the more complex form that will later be discusses.

Looking at this curve one can ask the question, what growth rate is implied. Let:

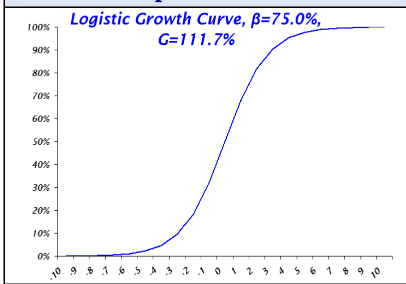
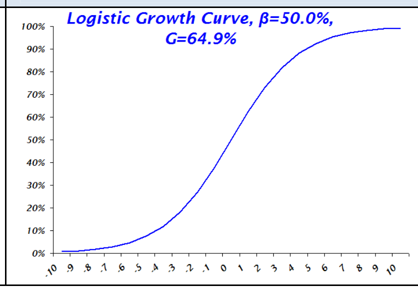
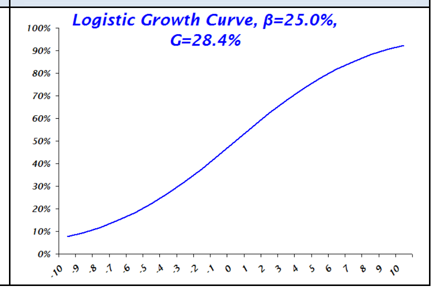
$$R_t = \frac{Y_t}{1-Y_t}$$

This is the ratio of the portion of the population that has adopted at time *t*, to the proportion that has not. While R_t will grow from a very small number to a very larger number, its growth will be constant over time and can be given by $G = e^{\beta} - 1$, where β = the growth/shape parameters. In the example above, β is implied to be equal to 1. Therefore, the adoption penetration rate in this case will be equal to 171.8%. Adding β, the logistics function produces:

$$Y_t = \frac{1}{1+e^{-\beta t}}$$

The graphs below show how the shape of the curve changes as the parameter β changes.

Logistics Curves

Rapid Growth	Moderate Growth	Slow Growth
		

As can be seen in the above graphs, as β increases the curve becomes flatter and the portion of the curve that spans the most growth becomes wider. β is best interpreted as the shape parameter rather than as a growth parameter even though it's value can be derived as a function of growth metrics as will be shown in subsequent discussions and is related to constant rate of change in the adoption to non-adoption ratio *G*.

While the addition of the β parameter has provided a way to manage the shape of the curve, it still lacks a method to control its location within a fixed period. Additionally, it would be more intuitive if the time variable value could be set such that the first period of actual sales is at *t*=1, or *t*=0. Both of these require the addition of another parameter α that will control the location of the curve relative to a set time framework.

From the discussion above it should be clear that the change in penetration per time period at any time period would be proportional to both the penetration to date, the percent left, and a growth constant. This may be expressed as the differential equation:

$$\frac{dy}{dt} = \beta \cdot Y \cdot (1 - Y)$$

Defining α as the value of t where Y = 50% then we can integrate the above equation between *t=0* and *t=α*.

$$\int_{t=0}^{t=\alpha} \beta \cdot Y \cdot (1-Y)dt$$

There are a number of closed from solutions. The most common are:

Internal Life Cycle Models

Mansfield-Blackman	Pearl-Reed	Fisher-Pry
$Y_t = \frac{1}{1+e^{-(\alpha+\beta t)}}$ **or** $Y_t = \frac{1}{1+e^{(\alpha-\beta t)}}$	$Y_t = \frac{1}{1+\alpha e^{-\beta t}}$	$Y_t = \frac{1}{1+e^{-b(t-\alpha)}}$

Mathematically these functions are the same in that one may derive any of the others. For example, the Fisher-Pry equation may be derived from the Pearl-Reed equations as follow: Let α_p = the location parameter for the Pearl Curve and α_F = the location parameter for the Fisher-Pry Curve. Then if $\alpha_p = e^{\alpha_F \beta}$

$$Y_t = \frac{1}{1+\alpha_p e^{-\beta t}} = \frac{1}{1+e^{\alpha_F \beta} \cdot e^{-\beta t}} = \frac{1}{1+e^{\alpha_F \beta - \beta t}} = \frac{1}{1+e^{-\beta(t-\alpha_F)}}$$

Furthermore, given the same set of historical data from which to estimates the parameters, the resulting models would produce the same historical fit and forecast. They differ only in terms of the parameter formulations.

The logistics function may also be recast as a probability distribution function:

The Logistics Distribution

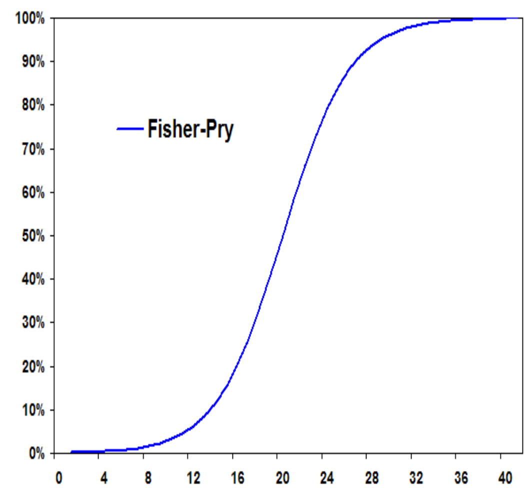
$$Y_t = \frac{1}{1+e^{-\left(\frac{t-\mu}{\sigma}\right)}}$$

where μ is the mean that replaces α as the location parameter, and σ is the standard deviation that replaces β as the shape parameter.   Note: given the same mean and standard deviation the logistics distribution will produce a flatter curve than the normal distribution.

Model Equivalency and Selection - In the prior discussion we set *G* = to the constant rate of change in the ratio of adopters to non-adopters. We also showed that *G* can be derived from β and therefore by reversing the equation, $\beta = \ln(G+1)$.   Another property of the logistics curve is that growth rate of the penetration function itself *Y*, will be equal to half of *G* at exactly the point where *Y* = 50%. This is also the point where t = *α*. In other words, these functions are all symmetrical around the point t = *α* which is the point at which half of the population has been penetrated.

Therefore in order for the instantaneous growth of Y to be 20% at the mid-point of the curve then the shape parameter will be $\beta = Ln(2i+1)$ = 33.6%. If we also wish the mid-point to be the 20th period, we need to find the value of *α* that will produce this result. The Fisher-Pry function is the only one of the functions listed above that allows both the location parameter and the shape parameter to be directly set to the desired values. In this case we can set the Fisher-Pry parameters to β = 33.6% and the location parameter to be set to α = 20 and produce the logistics curve shown to the right.

Fisher-Pry Curve



In order for the other function to produce the same curve the input parameters need to be transformed as shown below

Life Cycle Curve Parameters

Model	Shape/Growth Parameter	Location Parameter
Logistic Distribution	$\beta = \frac{1}{\hat{\beta}}$ (Standard Deviation)	$\alpha = \hat{\alpha}$ (mean)
Mansfield-Blackman	$\beta = \hat{\beta}$	$\alpha = -(\beta \cdot \hat{\alpha})$
Fisher-Pry	$\beta = \hat{\beta}$	$\alpha = \hat{\alpha}$
Pearl-Reed	$\beta = \hat{\beta}$	$\alpha = e^{\beta} \cdot \hat{\alpha}$

It is obvious that the adoption model that is most intuitive, and whose parameter formulations are closest to common management metrics, is the Fisher-Pry.

Influence Models - The Fisher-Pry model is one of several that are classified in the literature as a model of Internal or External Influence. The complete taxonomy is:

Life Cycle Influence Models

Internal Influence	External Influence	Mixed Influence Bass Diffusion Model
$\frac{dy}{dt} = q \cdot Y \cdot (1 - Y)$	$\frac{dy}{dt} = p \cdot (1 - Y)$	$\frac{dy}{dt} = [p + qY] \cdot (1 - Y)$

Note that in the Internal Influence case the growth parameter β has been replaced with the parameter *q* that is referred to as the coefficient of Internal Influence. The mathematics have not changed, only the conceptual interpretation. The parameter *p* in the External Influence equation is referred to as the coefficient of External Influence. The Mixed Influence classification included both parameters. These models frame the adoption process in terms of the method of communication.

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Daniel Research Group offers consulting and market research services to clients whose products and services are technology based or enabled. The primary focus is on providing results, solutions, consulting and training to clients that have strategic and tactical decisions that require Forecast, Segmentation, Market Share, and other market modeling requirements These engagements are supported with the full range of traditional market research data gathering and analysis services, including quantitative and qualitative surveys, focus groups, demographic and firmographic data acquisition and analysis, as well as input from technology and industry experts. While our emphasis is on delivering data and actionable recommendations, we often design and develop custom models and modeling tools for client use as well as providing training in these areas.

Stephen J. Daniel - President

Mr. Daniel's three decades in the Information Technology Industry has given him a unique blend of Market and Technology experience coupled with a deep understanding of Market Research Methodology. His primary strength is in understanding the decision making context within which the results of his research will be applied. This is manifested by his ability to design and execute studies that precisely meet client objectives on schedule at reasonable costs.

After receiving his BS in Finance in 1970 from Northeastern University, Mr. Daniel earned an MBA in Quantitative Analysis from New York University in 1974. He is a member of the American Statistical Association, The Market Research Association of America, the American Marketing Association and the Qualitative Research Association of America.

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